EXTREMAL PROBLEMS AND THE DIAMETER OF DELETED GRAPHS

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Abstract. For integers \( d \geq 1 \) and \( m \geq 0 \) let \( G_n \) be a graph of order \( n \), and for each \( m' \leq m \) let \( v_i, 1 \leq i \leq m' \), and \( e_i, 1 \leq i \leq m' \), be arbitrary collections of \( m' \leq m \) vertices and edges of \( G_n \) respectively. Let \( D_{d,m} \) be the property that for each \( m' \leq m \) the graph \( G_n - \{v_1, v_2, \ldots, v_m\} \) has a path of length at most \( d \) between each pair of vertices, and let \( D'_{d,m} \) be the property that for each \( m' \leq m \) the graph \( G_n - \{e_1, e_2, \ldots, e_m\} \) has a path of length at most \( d \) between each pair of vertices. The problem of determining the minimum number of edges in a graph with property \( D_{d,m} \) or the corresponding edge property \( D'_{d,m} \) as well as the structure of such extremal graphs will be considered. For some specific \( d \) and \( m \) precise results will be obtained and bounds will be given in other cases.

1. INTRODUCTION

Consider a graph \( G_n \) which represents a computer network with the vertices corresponding to processors and the edges corresponding to communication links. Thus processor failures or link failures in the network correspond to vertex and edge deletions in the graph \( G_n \). We want to insure that even after some vertex deletions or edge deletions (processor failures or link failures) that there is a path of reasonable length between each pair of vertices. Furthermore, we want \( G_n \) to be of minimal size (number of edges). We formalize these concepts with the following definitions.

Definition 1. Let \( d \geq 1 \) and \( m \geq 0 \) be integers, \( G_n \) be a graph of order \( n \), and for each \( m' \leq m \), let \( \{v_i | 1 \leq i \leq m'\} \) and \( \{e_i | 1 \leq i \leq m'\} \) be arbitrary collections of \( m' \) vertices and edges of \( G_n \) respectively. Let \( D_{d,m} \) be the property that for each \( m' \leq m \) the graph \( G_n - \{v_1, v_2, \ldots, v_m\} \) has a path of length at most \( d \) between each pair of vertices, and let

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Let $D'_{d,m}$ be the corresponding edge property that the graph $G_n = \{e_1, e_2, ..., e_m\}$ has a path of length at most $d$ between each pair of vertices.

Both property $D_{d,m}$ and $D'_{d,m}$ imply that the diameter of the graph $G_n = \{v_1, v_2, ..., v_m\}$ and the graph $G_n = \{e_1, e_2, ..., e_m\}$ respectively is at most $d$. If $n \leq m + 1$, the only graph $G_n$ that satisfies $D_{d,m}$ is the complete graph $K_m$, and this is a consequence of the fact that less than $m$ vertices can be deleted in determining if a graph satisfies $D_{d,m}$. No graph, except $K_1$, satisfies $D'_{d,m}$ if $n \leq m + 1$, which is also a consequence of the fact that less than $m$ edges can be deleted in determining property $D'_{d,m}$. If we deal only with graphs with at least $m + 2$ vertices and $m$ edges, then Definition 1 would be equivalent to a corresponding definition that required the deletion of precisely $m$ vertices or $m$ edges. We will use this alternate definition when it is equivalent to Definition 1, since it will be more convenient.

**Definition 2.** For any graphical property $P$, let $\text{ext}_n(P)$ denote the minimum number of edges in a graph of order $n$ that has property $P$. The class of graphs of order $n$ with $\text{ext}_n(P)$ edges that have property $P$ will be denoted by $\text{EXT}_n(P)$.

Our objective is to study the extremal numbers $\text{ext}_n(D_{d,m})$ and $\text{ext}_n(D'_{d,m})$, and the extremal classes $\text{EXT}_n(D_{d,m})$ and $\text{EXT}_n(D'_{d,m})$. For some values of $d$ and $m$ we will give exact values for $\text{ext}_n(D_{d,m})$ and $\text{ext}_n(D'_{d,m})$, and determine the extremal classes $\text{EXT}_n(D_{d,m})$ and $\text{EXT}_n(D'_{d,m})$. In other cases only bounds will be given.

Related to the properties $D_{d,m}$ and $D'_{d,m}$ are the stronger Menger Path System properties $P_{d,m}$ and $P'_{d,m}$, which have been studied in [3], [4], [5], and [6]. A graph $G_n$ satisfies property $P_{d,m}$ (respectively $P'_{d,m}$) if there are $m$ vertex (respectively edge) disjoint paths of length at most $d$ between each pair of vertices in the graph $G_n$. Clearly property $P_{d,m+1}$ implies $D_{d,m}$ and $P'_{d,m+1}$, and property $P'_{d,m+1}$ implies $D'_{d,m}$, and therefore there are the following inequalities:

$$\text{ext}_n(D_{d,m}) \leq \text{ext}_n(P_{d,m+1})$$

$$\text{ext}_n(D'_{d,m}) \leq \text{ext}_n(P'_{d,m+1})$$

Results on extremal properties for Menger Paths Systems can be found in [2] and will be used in this paper.

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The notation will generally follow [1]. Some special notation is similar to that used in [2], but we will define any special notation as needed.

2. \(d \geq 4\)

Any graph \(G_n\) that satisfies the property \(D_{d,m}\) (or \(D'_{d,m}\)) must be \((m+1)\)-connected \(((m+1)\)-edge connected), and so each vertex of \(G_n\) must be of degree at least \(m+1\). Thus, it is clear that in general

\[
\text{ext}_n(D_{d,m}), \text{ext}_n(D'_{d,m}) \geq n(m+1)/2.
\]

For small \(n\) there are graphs that are regular of degree \(m+1\) (or almost regular if \(n\) and \(m+1\) are both odd) that satisfy both \(D_{d,m}\) and \(D'_{d,m}\). We now describe two families of graphs with both of these properties. In the case when \(m\) is odd and \(n < 2m+2\), the power of a cycle \(C_n^{(m+1)/2}\) is regular of degree \(m+1\) and can easily be shown to satisfy both \(D_{d,m}\) and \(D'_{d,m}\). Also, when \(m\) is even, the graph \(C_n^{m/2}\) with the longest chords added is almost regular of degree \(m+1\) and can also be shown to satisfy both \(D_{d,m}\) and \(D'_{d,m}\). For the second family consider the cartesian product \(H_{d,m} = K_m \times C_d\), which is a graph of order \(n = dm\) that contains \(d\) disjoint copies of the complete graph \(K_m\) and a complete matching between the vertices of the pairs of complete graphs that represent adjacent vertices on the cycle \(C_d\). It is easy to verify that \(H_{d,m}\) satisfies both \(D_{d,m}\) and \(D'_{d,m}\). The graph \(H_{d,m} = K_m \times C_d\) can be modified by replacing two of the adjacent complete graphs with an appropriate power of a cycle to give a graph satisfying \(D_{d,m}\) and \(D'_{d,m}\) when \(n \leq dm\) but not divisible by \(m\). These facts are summarized in the following result.

**THEOREM 1.** For \(d \geq 4\) and \(m+1 < n \leq dm\),

\[
\text{ext}_n(D_{d,m}) = \text{ext}_n(D'_{d,m}) = \lfloor n(m+1)/2 \rfloor.
\]

We now consider graphs \(G_n\) that satisfy properties \(D_{d,m}\) and \(D'_{d,m}\) for \(n\) large. For \(d\) and \(m\) fixed, and \(n\) sufficiently large it is clearly not possible for an \((m+1)\)-regular graph
of order $n$ to satisfy $D_{d,m}$ and $D'_{d,m}$, since the diameter of the graph would depend upon $n$, and thus would not be bounded. In fact, the number of vertices within a distance $d$ of a fixed vertex in a graph of maximal degree $m+1$ is at most

$$1 + (m + 1)(1 + m + \cdots + m^{d-1}) \leq (m + 1)^d.$$ 

Therefore, if $n > (m + 1)^d$, no graph of order $n$ with maximum degree $m + 1$ can satisfy $D_{d,m}$ or $D'_{d,m}$.

Next we will show that any graph of minimal degree at least $m + 1$ and diameter $d$ must have at least $n(m + 1)/2 + |n/(2m^d)|$ edges. Consider any graph $G_n$ that has diameter $d$ and minimum degree $m + 1$. If each vertex of $G_n$ has degree $m + 2$, then $G_n$ has at least $n(m + 2)/2 = n(m + 1)/2 + n/2$ edges. If there is a vertex of degree $v$ of degree $m + 1$, then there is a spanning tree $T_n$ of $G_n$ rooted at $v$ that has height at most $d$. Prune this tree, starting with the vertices at the greatest distance from $v$, by deleting edges from those vertices of degree exceeding $m + 1$ until their degree is precisely $m + 1$. This pruned tree $T'$ will have at most $(m + 1)^d$ vertices, and also when an edge is pruned from a vertex of degree greater than $m + 1$, less than $m^{d-1}$ other edges will be deleted as a result of this pruned edge. Therefore, for $n$ sufficiently large, at least

$$\left(n - 1 - (m + 1)^d\right)/m^{d-1} \geq n/m^d$$

edges were deleted from vertices to reduce their degree to $m + 1$. Hence, $G_n$ has at least $n(m + 1)/2 + n/(2m^d)$ edges.

We will give examples that can be found in [2], which will give an upper bound for $\text{ex}_n(D_{d,m})$ and $\text{ex}_n(D'_{d,m})$. In fact, for $d = 2k + 2$, $k \geq 1$, we will describe a graph $G_n$ that satisfies $D_{d,m}$ and $D'_{d,m}$, and has $n(m + 1)/2 + (n - m - 1)/2k$ edges. For convenience we will assume that $n = (m + 1)(k + 1)$ for some integer $k$. First, consider the case of $k = 1$.

The cartesian product $K_{m+1} \times K_{1,t}$ has all of its vertices of degree $m + 1$, except for $m + 1$ vertices associated with the center of the star, which have degree $m + t$. It is straightforward to show that this graph satisfies both $D_{d,m}$ and $D'_{d,m}$. Let $S_{t,k}$ denote the graph with $kt + 1$ vertices obtained from $K_{1,t}$ by subdividing each of the edges $k - 1$ times. Thus, $S_{t,k}$ consists
of \( \ell \) paths of length \( k \) with one common end vertex. The graph \( K_{m+1} \times S_{\ell,k} \) can be shown to satisfy both \( D_{d,m} \) and \( D'_{d,m} \) for \( d = 2k + 2 \). In fact, a perfect matching can be deleted from each of the copies of the complete graph \( K_{m+1} \), except for the complete graph associated with the center of the star and the last two complete graphs in each of the \( \ell \) paths, to obtain a graph \( H_{n,k,m} \) that still satisfies both \( D_{d,m} \) and \( D'_{d,m} \). It can easily be checked that \( H_{n,k,m} \) has all of its vertices of degree \( m + 1 \) except for \( m + 1 \) vertices (those vertices associated with the center of the star) of degree \( m + \ell \) and \( \ell(m + 1) \) vertices (those vertices associated with the next to last vertex on each of the paths) of degree \( m + 2 \). Minor modifications of the graphs just described can be made to obtain similar graphs to deal with the cases when \( d \) is not even and \( n \) does not satisfy the divisibility conditions assumed above.

The following summarizes the results just discussed, and shows that for \( d \geq 4 \) we have the correct order of magnitude for both \( \text{ext}_n(D_{d,m}) \) and \( \text{ext}_n(D'_{d,m}) \).

**Theorem 2.** For \( d \geq 4 \) and \( n \) sufficiently large

\[
\frac{n(m + 1)}{2} + n/2m^d \leq \text{ext}_n(D_{d,m}), \text{ext}_n(D'_{d,m}) \leq \frac{n(m + 1)}{2} + n/(2|d/2| - 2).
\]

3. \( d = 1 \) or \( 2 \)

For \( d = 1 \) the extremal problem is trivial and not interesting. Any pair of vertices would have to be adjacent in any graph that satisfied \( D_{1,m} \), so \( \text{ext}_n(D_{1,m}) = \binom{n}{2} \) and \( \text{EXT}_n(D_{1,m}) = \{K_n\} \). No graph satisfies \( D'_{1,m} \) for \( m \geq 1 \), since the edge between any pair of vertices could be deleted. A graph would have to be complete to satisfy \( D'_{1,0} \), and so \( \text{ext}_n(D'_{1,0}) = \binom{n}{2} \) and \( \text{EXT}_n(D'_{1,0}) = \{K_n\} \).

We will first consider property \( D_{d,m} \). For \( d = 2 \) and \( n > 4m + 2 \) the following result gives the extremal numbers and the extremal graphs for \( D_{2,m} \).

**Theorem 3.** For any \( m \geq 0 \) and \( n > 4m + 2 \)

\[
\text{ext}_n(D_{2,m}) = (m + 1)(n - m - 1), \text{ and } \text{EXT}_n(D_{2,m}) = \{K_{m+1, n-m-1}\}.
\]

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PROOF: It should first be noted that for \( n > 4m + 2 \) the complete bipartite graph \( K_{m+1,n-m-1} \) satisfies \( D_{2,m} \), since the deletion of any \( m \) vertices will always leave a complete bipartite graph with vertices in each part. Any complete bipartite graph has diameter at most 2.

It remains to show that any graph \( G_n \) with \( n > 4m + 2 \) that satisfies \( D_{2,m} \) has at least \((m+1)(n-m-1)\) edges with equality if and only if \( G_n \) is isomorphic to \( K_{m+1,n-m-1} \). If each vertex of \( G_n \) has degree at least \( 2m+2 \), then \( G_n \) has \((m+1)n\) edges and this case is complete. Thus, we can select a vertex \( v \) of \( G_n \) of minimum degree \( k \) with \( m + 1 \leq k < 2m + 2 \). Let \( N \) be the neighborhood of \( v \) and \( M \) be the remaining \( n - k - 1 \) vertices. Each vertex of \( M \) must be adjacent to at least \( m + 1 \) vertices of \( N \) for \( G \) to satisfy \( D_{2,m} \), so there are \((m+1)(n-k-1)\) edges between \( M \) and \( N \). Also, note that each vertex of \( M \) has degree at least \( k \), so there are at least

\[
k + (m+1)(n-k-1) + \frac{(k-m-1)(n-k-1)}{2} = k + \frac{n-k-1}{2}(k+m+1)
\]

edges in \( G_n \). It is sufficient to show that the function

\[
f(k) = k + \frac{n-k-1}{2}(k+m+1) - (m+1)(n-m-1) = \frac{k-m-1}{2}(n-2m-k-1)
\]

is not negative. However, it is clear that \( f(k) \geq 0 \) for \( m + 1 \leq k \leq 2m + 1 \). In addition, \( f(k) = 0 \) if and only if \( k = m + 1 \), so the graph \( G_n \) will have precisely \((m+1)(n-m-1)\) edges when \( k = m + 1 \) and each vertex of \( M \) has degree \( m + 1 \). This is the case when \( G_n \) is isomorphic to \( K_{m+1,n-m-1} \). This completes the proof of Theorem 3. ■

Using the counting techniques of the proof of Theorem 3, it is straightforward to verify that \( ext_n(D_{2,m}) = (m+1)(n-m-1) \) and \( ext_n(D_{2,m}) = \{K_{m+1,n-m-1}\} \) for \( n = 2m + 2, 2m + 3, \) and \( 2m + 4 \). These results indicate that the conclusions of Theorem 2 might be valid for all \( n \geq 2(m+1) \). For small values of \( m \) this can be verified. Thus, we make the following conjecture.
CONJECTURE. For any \( m \geq 0 \) and \( n \geq 2m + 2 \)

\[
\text{Ext}_n(D_{2,m}) = (m+1)(n-m-1), \quad \text{and}
\]

\[
\text{Ext}_n(D_{2,m}) = \{K_{m+1,n-m-1}\}.
\]

Next we investigate \( \text{Ext}_n(D_{2,m}) \) for \( n \leq 2(m+1) \). At the extremes we have trivially, \( \text{Ext}_{2m+2}(D_{2,m}) = \binom{m+2}{2} \) and \( \text{Ext}_{2m+2}(D_{2,m}) = \{K_{m+3}\} \), and \( \text{Ext}_{2m+2}(D_{2,m}) = (m+1)^2 \) and \( \text{Ext}_{2m+2}(D_{2,m}) = \{K_{m+1,m+1}\} \). Both of these extremal graphs are regular of degree \( m + 1 \), and it is also possible that \( \text{Ext}_n(D_{2,m}) = n(m+1)/2 \) for other values of \( n \) as well.

Consider the graph \( H = K_{m+1+t} - (K_t \cup K_{t_2} \cup \cdots \cup K_{t_t}) \), which is the graph obtained from the complete graph \( K_{m+1+t} \) by deleting edges from vertex disjoint copies of complete graphs of the indicated orders. Clearly \( H \) is a complete multipartite graph, and any graph \( H_{t+1} \) obtained from \( H \) by deleting \( m \) vertices is also a complete multipartite graph. If, in addition, \( t_i \leq t \) for all \( i \), then the complete multipartite graph \( H_{t+1} \) has at least \( 2 \) parts, and hence has diameter at most \( 2 \). This notation can be used to describe a family of extremal graphs.

Suppose \( n = m+1+t \) with \( 1 \leq t \leq m+1 \), and \( n = tq + r \) with \( 0 \leq r < t \). Then, consider the graph

\[
H = K_{m+1+t} - (qK_t \cup K_r),
\]

which satisfies \( D_{2,m} \) and has

\[
tq(m+1) + r(m+1+t-r) = \frac{n(m+1)}{2} + \frac{r(t-r)}{2}
\]

edges. Thus, when \( r = 0 \) (i.e. \( t \) divides \( m+1 \)) the graph \( H \) is regular of degree \( m+1 \) and \( H \) is in \( \text{Ext}_{m+1+t}(D_{2,m}) \). Also, when \( r \) is small (or, in particular, when \( t \) is small) compared to \( n \), then the number of edges in \( H \) is approximately \( n(m+1)/2 \), which is the minimum number possible.

It can be verified for small values of \( m \) that the family of graphs described in Theorem 3 and in the discussion following it give the extremal numbers for \( \text{Ext}_n(D_{2,m}) \). For \( m = 0, 1, 2, \)
and 3 and \( n = m + 1 + t \) with \( 1 \leq t \leq m + 1 \), direction calculation shows that the number of edges in the graph \( H \) (i.e. \( n(m+1)/2 + r(t-r)/2 \)) is the extremal number \( \text{ext}_{m+1+1}(D_{2,m}) \). Also, for \( m = 0, 1, 2, \) and 3 the counting techniques used in the proof of Theorem 3 can be used to show that \( \text{ext}_n(D_{2,m}) = (m + 1)(n - m - 1) \) and \( \text{EXT}_n(D_{2,m}) = \{K_{m+1,n-m-1}\} \) for \( n \geq 2m + 2 \). Thus, the extremal graphs of order \( n \) that satisfy \( D_{2,m} \) for small values of \( m \) are the following:

\[
m = 0, \\
K_{1,n-1} \text{ for } n \geq 1,
\]

\[
m = 1, \\
K_1, K_2, K_3, \text{ and } K_{2,n-2} \text{ for } n \geq 4,
\]

\[
m = 2, \\
K_1, K_2, K_3, K_4, K_5 - 2K_2, \text{ and } K_{3,n-3} \text{ for } n \geq 6,
\]

\[
m = 3, \\
K_1, K_2, K_3, K_4, K_5, K_6 - 3K_2, K_7 - 2K_3, \text{ and } K_{4,n-4} \text{ for } n \geq 8.
\]

For \( d = 2 \) and \( n \) large the nature of the extremal graphs for \( D'_{2,m} \) is similar to those for \( D_{2,m} \) but not identical. Since edge deletion leaves the incident vertices, \( D'_{2,m} \) implies more edges in the extremal graphs than \( D_{2,m} \). For graphs of large order \( n \) the following result gives the extremal numbers and the extremal graphs for \( D'_{2,m} \), and is the result that corresponds to Theorem 3 for \( D_{2,m} \).

**Theorem 4.** For any \( m \geq 0 \) and \( n > 3m + 2 \)

\[
\text{ext}_n(D'_{2,m}) = (m + 1)(n - m - 1) + \binom{m+1}{2}, \text{ and}
\]

\[
\text{EXT}_n(D'_{2,m}) = \{K_{m+1} + K_{n-m-1}\}.
\]

**Proof:** The graph \( K_{m+1} + K_{n-m-1} \) has \( (m+1)(n-m-1) + m(m+1)/2 \) edges, and it is straightforward to verify that it satisfies \( D'_{2,m} \). To complete the proof of Theorem 4 it is sufficient to show that any graph \( G_n \) of order \( n > 3m + 2 \) that satisfies \( D'_{2,m} \) has at
least \((m + 1)(n - m - 1) + m(m + 1)/2\) edges, with equality only if it is isomorphic to \(K_m + \overline{K}_{n-m}\).

If each vertex of \(G_n\) has degree at least \(2m + 2\), then \(G_n\) has \((m + 1)n > (m + 1)(n - m - 1) + m(m + 1)/2\) edges. Therefore, we can assume there is a vertex of degree less than \(2m + 2\).

Select a vertex \(v\) of \(G_n\) of minimum degree \(k\) with \(m + 1 \geq k < 2m + 2\). Let \(N\) be the neighborhood of \(v\) and \(M\) be the remaining \(n - k - 1\) vertices. Each vertex of \(N\) must be adjacent to at least \(m\) other vertices of \(N\) for there to be a path of length at most 2 between \(v\) and each of the vertices in \(N\) after the deletion of \(m\) edges of \(G\). Thus, there are at least \(km/2\) edges in \(N\). Also, each vertex of \(M\) must be adjacent to at least \(m + 1\) vertices of \(N\) for \(G\) to satisfy \(D'_{2,m}\), so there are \((m + 1)(n - k - 1)\) edges between \(M\) and \(N\). Finally, note that each vertex of \(M\) has degree at least \(k\), so that are at least

\[
k + (m + 1)(n - k - 1) + \frac{km}{2} + \frac{(k - m - 1)(n - k - 1)}{2}
\]

edges in \(G_n\). It is sufficient to show that the function

\[
f(k) = k + \frac{km}{2} + \frac{n - k - 1}{2}(k + m + 1) - (m + 1)(n - m - 1) - \binom{m + 1}{2} = \frac{k - m - 1}{2}(n - m - k - 1)
\]

is not negative. However, it is clear that \(f(k) \geq 0\) for \(m + 1 \leq k \leq 2m + 1\). In addition, \(f(k) = 0\) if and only if \(k = m + 1\), so the graph \(G_n\) will have precisely \((m + 1)(n - m - 1) + \binom{m + 1}{2}\) edges precisely when \(k = m + 1\), each vertex of \(M\) has degree \(m + 1\), and each vertex of \(N\) is adjacent to all of the other vertices of \(N\). This is the case when \(G_n\) is isomorphic to \(K_{m+1} + \overline{K}_{n-m-1}\). This completes the proof of Theorem 4.

Next we investigate \(e(D'_{2,m})\) for \(n \leq 2m + 1\). If \(G_n\) is a graph that satisfies \(D'_{2,m}\), then note that any two adjacent vertices in \(G_n\) must have \(m\) common adjacencies, and any pair of nonadjacent vertices must have \(m + 1\) common adjacencies; for otherwise, the deletion of \(m\) edges would leave no path of length at most 2 between the vertices. In fact,
the converse is also true, and so if these neighborhood intersection conditions are satisfied for each pair of vertices of a graph, then the graph will satisfy $D'_{2,m}$. Any graph $G_n$ of minimal degree at least $(n + m)/2$, satisfies the neighborhood intersection conditions, and therefore satisfies $D'_{2,m}$.

We now have two families of graphs that satisfy $D'_{2,m}$, namely: $K_{m+1} + \overline{K}_{n-m-1}$ which has $f(n) = (m + 1)(2n - m - 2)/2$ edges, and all nearly regular (all vertices have the same degree except when the degree of regularity and the order of the graph are odd and then one vertex has degree one larger) graphs of degree $\lceil \frac{n + m}{2} \rceil$ which have $g(n) = \lceil \frac{m + n}{2} \rceil$ edges. This implies that

$$\exists \lambda_n(D_{2,m}) \leq \min \{f(n), g(n)\}.$$ 

Note that $f(m + 2) = g(m + 2) = \binom{m+2}{2}$ and $f(2m + 2) = g(2m + 2) = \frac{(m+1)(3m+2)}{2}$. In the case when both $n$ and $m$ are even,

$$f(n) - g(n) = \frac{1}{4}(n - (m + 2))(2m + 2 - n) \geq 0$$

for $m + 2 \leq n \leq 2m + 2$. The same inequality is valid for the other parities of $n$ and $m$ with a few exceptions. Therefore the nearly regular graphs generally give a smaller lower bound for $\exists \lambda_n(D_{2,m})$ in this interval. However, the bound given by $g(n)$ is not sharp over this entire interval. This problem is discussed in [2], since $D'_{2,m} = P_{2,m} + 1$, and in that paper graphs that come from finite geometries are shown to give better upper bounds for $P_{2,m+1}$.

For small values of $m$ the extremal numbers $\exists \lambda_n(D_{2,m})$ and the extremal graphs $\text{EXT}_n(D'_{2,m})$ can be determined. It is straightforward to verify that

$$\exists \lambda_{m+3}(D_{2,m}) = \binom{m+3}{2}$$

and $\text{EXT}_{m+3}(D_{2,m}) = \{K_{m+2}\}$,

$$\exists \lambda_{m+4}(D_{2,m}) = \binom{m+4}{2} - 1$$

and $\text{EXT}_{m+4}(D_{2,m}) = \{K_{m+2} - K_2\}$,

$$\exists \lambda_{m+4}(D'_{2,m}) = \frac{m+4}{2} - \left\lfloor \frac{m+4}{2} \right\rfloor$$

and $\text{EXT}_{m+4}(D'_{2,m}) = \{K_{m+4} - M\}$,

where $M$ is a maximal matching in $K_{m+4}$ and $m \geq 4$. Extremal graphs of order $n$ that satisfy $D'_{2,m}$ for small values of $m$ are the following:

$m = 0,$

$K_{1,n-1}$ for $n \geq 1$, 

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\[ m = 1, \]
\[ K_2 + \overline{K}_{n-2} \text{ for } n \geq 3, \]
\[ m = 2, \]
\[ K_4, K_5 - K_2, K_6 - 3K_2, \text{ and } K_3 + \overline{K}_{n-3} \text{ for } n \geq 6, \]
\[ m = 3, \]
\[ K_5, K_6 - K_2, K_7 - 3K_2, K_8 - C_4 - 2K_2, P_{10}, \text{ and } K_4 + \overline{K}_{n-4} \text{ for } n \geq 7, \]

where \( P_{10} \) is the Petersen graph.

4. \( d = 3 \)

For \( n \geq m + 2 \), the graph \( K_{m+1,n-m-1} \) satisfies \( D_{2,m} \), so it certainly satisfies \( D_{3,m} \). Therefore, we have the inequality \( \text{ext}_n(D_{3,m}) \leq (m + 1)(n - m - 1) \). However, this bound is not sharp for \( d = 3 \) and \( n \) large as it was for \( d = 2 \).

Observe that if any graph \( G_n \) satisfies \( D_{3,m} \), then the graph \( G_{n+1} \) obtained from \( G_n \) by duplicating a vertex (replacing one vertex with a pair of independent vertices, each with the same neighborhood as the original vertex) also satisfies \( D_{3,m} \). Therefore, since \( C_5 \) is in \( \text{EXT}_3(D_{3,1}) \), a graph \( G_n \) with \( 2n - 5 \) edges that satisfies \( D_{3,1} \) can be generated by duplicating vertices of the \( C_6 \). Likewise, the graph \( C_5' \), which is a cycle \( C_6 \) with two long chords added between the first and fifth and the third and seventh vertices, satisfies \( D_{3,1} \). Duplicating the vertices of degree 2 in this graph will generate a graph \( G_n \) with \( 2n - 6 \) vertices that satisfies \( D_{3,1} \). Both of these families of graphs have less than the \( 2n - 4 \) edges in the family of complete bipartite graphs \( K_{2,n-2} \). In the same fashion the graph \( C_6' \), which is a \( C_8 \) with all the long chords added, satisfies \( D_{3,2} \), and can have its vertices duplicated to obtain a graph \( G_n \) with \( 3n - 12 \) edges that satisfies \( D_{3,2} \). This is less that the \( 3n - 9 \) edges in \( K_{3,n-3} \). In general, it is not difficult to find graphs for any \( m \) that satisfy \( D_{3,m} \) and have less than \( (m + 1)(n - m - 1) \) edges. However, we have no examples that are significantly less. Later, a lower bound for \( \text{ext}_n(D_{3,m}) \) will be given in Theorem 5.

For \( n = 1 + 2t \), the graph \( K_1 + tK_2 \) ("friendship" graph) can be shown to satisfy \( D_{2,1}^t \), and so in this case \( \text{ext}_n(D_{2,1}^t) \leq 3(n - 1)/2 \) edges. For even \( n \) a vertex of degree 2 can be
duplicated to get a graph that satisfies \( D'_{3,1} \), so for arbitrary \( n \) we have

\[
\text{ext}_n(D'_{3,1}) \leq \left\lfloor \frac{3(n-1)}{2} \right\rfloor.
\]

Later in this section Theorem 5, which will verify that \( 3(n-1)/2 \) is the correct order of magnitude for \( \text{ext}_n(D'_{3,1}) \), will be proved. The friendship graph can be generalized to give a lower bound example for arbitrary \( m \). For \( n = \left( \lfloor (m+1)/2 \rfloor + t\lfloor (m+1)/2 \rfloor \right) \), the graph \( K_{\lfloor (m+1)/2 \rfloor} + tK_{\lfloor (m+1)/2 \rfloor} \) can be shown to satisfy \( D'_{3,m} \). To obtain examples for arbitrary \( n \), this example can be modified by duplicating vertices of degree \( m+1 \). Therefore, we have the following inequality for \( n > m+2 \):

\[
\text{ext}_n(D'_{3,m}) \leq \frac{3m+4}{4} n.
\]

This is less than the corresponding extremal number \( (m+1)(n-m-1) \) for \( D'_{2,m} \) that results from a complete bipartite graph.

We now consider lower bounds for both \( \text{ext}_n(D'_{3,m}) \) and \( \text{ext}_n(D'_5,m) \). The following result gives such a bound.

**THEOREM 5.** For \( m \geq 0 \), \( n > m+3 \), and any given \( \epsilon > 0 \), there is a constant \( C \) such that

\[
\text{ext}_n(D'_{3,m}), \, \text{ext}_n(D'_5,m) > \left( \frac{m+2}{2} - \epsilon \right)n - C.
\]

**PROOF:** The proof will be by induction on the order \( n \) of a graph \( G_n \) satisfying either \( D_{3,m} \) or \( D'_5,m \). Appropriate choice of \( C \) gives the result for small values of \( n \). Also, if each vertex of \( G_n \) has degree at least \( m+2 \), then the result follows trivially. If a vertex \( v \) of \( G_n \) has as many as \( m+1 \) duplicates, then it is straightforward to show that the graph \( G_{n-1} = G_n - v \) also satisfies \( D_{3,m} \) or \( D'_5,m \). Each vertex of \( G_n \) has degree at least \( m+1 \); therefore if \( G_{n-1} \) satisfies the required inequality so will \( G_n \). In addition, if the number of vertices of \( G_n \) of degree \( m+1 \) is at most \( 2(m+C) \), then a count of degrees implies that \( G_n \) satisfies the required inequality. Hence, we can assume that \( G_n \) has at least \( 2(m+C) \) vertices of degree \( m+1 \), and there are no more than \( m+1 \) duplicates of any vertex of \( G_n \).

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Select the smallest integer $t$ such that $t > m^2/(2c)$, and such that there is a collection $X$ of independent vertices of degree $m + 1$ in $G_n$ and such that the set $Y$ of vertices of $G_n$ adjacent to vertices in $X$ has $t$ vertices. The existence of such sets of vertices follows from the number of vertices of degree $m + 1$, and the restriction on the number of duplicates in $G_n$. Consider a vertex $y$ of $Y$. There is a vertex $x$ of $X$ with neighborhood $N(x) = \{y = y_1, y_2, \ldots, y_m\}$. If either $\{y_2, y_3, \ldots, y_{m+1}\}$ or $\{x_2, x_3, \ldots, x_{m+1}\}$ are deleted from $G_n$, then each of the remaining vertices must be within a distance 2 of $y$, since any path to $x$ contains $y$. Let $Z_{1y}$ be the vertices that are a distance 1 from $y$, and $Z_{2y}$ the vertices that are a distance 2 from $y$ in this deleted graph. Also, there is a distance subtree rooted at $y$ of this deleted graph in which all of the vertices of $Z_{2y}$ have degree 1.

Two cases will be considered.

First consider the case when there is some $y$ in $Y$ such that there are as many as $(1 - 2c/m)n - m - 1$ vertices in $Z_{2y}$. Since the distance tree has at least $n - m - 1$ edges, and since each vertex in $Z_{2y}$ has degree at least $m + 1$, there are at least

$$n - m - 1 + m(1 - 2c/m)n - m(m + 1) > (m + 2)/2 - c/n - C$$

edges in $G_n$.

If the previous case does not occur, then $|Z_{1y}| \geq (2c/m)n$, and so each vertex in $Y$ has degree at least $(2c/m)n$. A count on the sum of degrees in $G_n$ gives that $G_n$ has at least

$$\frac{(n-t)(m+1) + t2c}{2} \geq \frac{m+2}{2}n - C$$

edges. This completes the proof of Theorem 5. $\blacksquare$

In the case of property $D_{3,1}^*$, the lower bound given in Theorem 5 is the same order of magnitude as the number of edges in the "friendship" graphs that satisfy $D_{3,1}^*$. Thus,
reasonably sharp bounds exists for this case. However, this is not true for the remaining cases \((m \geq 2)\). Thus, there are many interesting problems remaining in the \(d = 3\) case, and it is not clear what should be the conjectured values for \(\text{ext}_n(D_{3,m})\) and for \(\text{ext}_n(D'_{3,m})\).

5. REFERENCES


